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Е.И. Смирнов

ХАУСДОРФОВЫ СПЕКТРЫ И ПУЧКИ ЛОКАЛЬНО ВЫПУКЛЫХ ПРОСТРАНСТВ

В настоящей статье рассматриваются обобщения подготовительной теоремы Вейерштрасса и глобальной теоремы Вейерштрасса о делении для ростков голоморфных функций в точке n-мерного комплексного пространства. Автор формулирует глобальную теорему о делении в терминах существования и непрерывности линейного оператора.

Ключевые слова: глобальная теорема Вейерштрасса о делении, теорема о замкнутом графике, ростки голоморфных функций, Н-пространства.

E.I. Smirnov

HAUSDORFF SPECTRA AND SHEAVES OF LOCALLY CONVEX SPACES

In the present article generalisation of the preparatory theorem by Vejershtrass and the global theorem by Vejershtrass about division for sprouts of holomorphic functions in a point of n-dimensional complex space are considered. The author formulates the global theorem about division in terms of existence and a continuity of the linear operator.

Keywords: The global theorem by Vejershtrass about division, the theorem of the closed schedule, sprouts of holomorphic functions, H-space.

Let $\{\mathcal{S}_U, \rho_{UV}\}$ be a presheaf of abelian groups over a topological space $\mathcal{D}$, Ω a nonempty partially ordered set and $\mathfrak{F}$ an admissible class for Ω (we may assume without loss of generality that $\Omega = |\mathfrak{F}|$). Let us denote by $\hat{H}(\mathcal{S})$ a covariant functor from $\mathrm{Ord}\,\Omega$ to $\mathrm{Ord}\,\mathcal{U}$, where $\mathcal{U}$ is a base of open sets in $\mathcal{D}$, and by $\check{H}(\mathcal{S})$ a contravariant functor from $\mathrm{Ord}\,\mathcal{U}$ to the category of abelian groups so that an abelian group $\mathcal{S}_U$ is defined for each $U \in \mathcal{U}$ and a homomorphism $\rho_{UV} : \mathcal{S}_U \to \mathcal{S}_V$ is defined for each pair $U \subset V$. Then $H = \check{H}(\mathcal{S}) \circ \hat{H}(\mathcal{S})$ is a contravariant functor of the Hausdorff spectrum $\mathcal{X}(\mathcal{S}) = \{\mathcal{S}_{U_s}, \mathfrak{F}, \rho_{U_{s'}U_s}\}$, which we will call the *Hausdorff spectrum associated with the presheaf* $\{\mathcal{S}_U, \rho_{UV}\}$. Let X be the H-limit of the Hausdorff spectrum $\mathcal{X}(\mathcal{S})$ in the category of abelian groups and let

$$A = \bigcap_{F \in \mathfrak{F}} \bigcup_{s \in |F|} U_s \,.$$

Proposition 1. *Let $\mathcal{S}$ be the sheaf of germs of holomorphic functions on an open set $\mathcal{D} \subset \mathbb{C}^n$, associated with the presheaf $\{\mathcal{S}_U, \rho_{UV}\}$, and let $\mathcal{X}(\mathcal{S}) = \{\mathcal{S}_{U_s}, \mathfrak{F}, \rho_{U_{s'}U_s}\}$ be the associated true Hausdorff spectrum. Then the H-limit of the Hausdorff spectrum $\mathcal{X}(\mathcal{S})$ is isomorphic to the vector space of sections $\Gamma(A, \mathcal{S})$ of the sheaf $\mathcal{S}$ over the set A.*

Proof. By the conditions relating to $\{\mathcal{S}_U, \rho_{UV}\}$, we may put $\mathcal{S}_U = \Gamma(U, \mathcal{S})$ $(U \in \mathcal{U})$. Further, let

$$X = \overset{\longleftarrow}{\varinjlim_{\mathfrak{F}}} \rho_{U_{s'}U_s} \Gamma(U_s, \mathcal{S}) \,,$$

so that

$$X = \bigcup_{F \in \mathfrak{F}} \bigcap_{T \in F} \psi(V_F^T) \,.$$

If $x \in X$, there exists $F \in \mathfrak{F}$ such that $x \in \psi(V_F^T)$ $(T \in F)$, that is to say, there exists a selection

$$\xi(T) = (f_s^T)_{s \in |F|}$$

such that $\psi(f_s^T) = x$ for each $T \in F$. For any $U \in \mathcal{U}_z$ $(z \in \mathcal{D})$ the homomorphism ρ_{zU} : $\Gamma(U, \mathcal{S}) \to \mathcal{S}_z$ generates for $f \in \Gamma(U, \mathcal{S})$ the set of points

$$\rho_U(f) = \bigcup_{z \in U} \rho_{zU}(f) \subset \mathcal{S} \,,$$

therefore let us put

$$\rho_x^T = \bigcup_{s \in T} \rho_{U_s}(f_s^T) \,;$$

it is clear that ρ_x^T generates the section f^T on the open set $U_T = \bigcup_{s \in T} U_s$, since the correspondence

$$z \in U_T \overset{f^T}{\longmapsto} \rho_x^T \cap \mathcal{S}_z \subset \mathcal{S}$$

is single-valued and continuous. Moreover, if $\rho_{UV} : \rho_V(g) \mapsto \rho_U(f)$, then $\rho_U(f) \subset \rho_V(g)$, so let us put

$$\rho_x^\xi = \bigcup_{F^* \succ F} \bigcup_{\substack{s^* \in |F^*| \\ s \in T}} \rho_{U_{s^*}U_s}(\rho_{U_s}(f_s^T)) \,,$$

where necessarily

$$\rho_{U_{s^*}U_s}(\rho_{U_s}(f_s^T)) = \rho_{U_{s^*}U_s}(\rho_{U_s}(f_s^{T'})) \quad (T, T' \in F) \,.$$

Let us put

$$U_{\rho_x} = \bigcap_{\xi} U_{\rho_x^\xi} \,, \quad \text{where} \quad U_{\rho_x^\xi} \subset \bigcup_{s \in |F|} U_s \,;$$

in this connection we have in particular,

$$\rho_{U_s}(f_s^T) \cap \rho_{U_s}(f_s^{T'}) \supset \rho_{U_{s^*}U_s}(\rho_{U_s}(f_s^T)) \,.$$

It is also clear that for each ξ the correspondence

$$z \in U_{\rho_x^\xi} \mapsto \rho_x^\xi \cap \mathcal{S}_z$$

is single-valued and continuous. Although, in general, it is not guaranteed that $U_{\rho_x} \neq \emptyset$, we will show nevertheless that $U_{\rho_x} \supset A$ under the conditions of the proposition, specifically because the H-limit of the Hausdorff spectrum $\mathcal{X}(\mathcal{S})$ is true. Let the selection $\xi(T) = (f_s^T)_{s\in|F|}$ $(T \in F)$ generating the element $x \in X$ be fixed. Then because the Hausdorff spectrum $\mathcal{X}(\mathcal{S})$ is true we may assume that $f_s^{T_1} = f_s^{T_2}$ $(s \in T_1 \cap T_2)$ and, consequently, there exists $\xi = (f_s)_{s\in|F|} \in \bigcap_{T\in F} V_F^T$ such that

$$x \in \psi((f_s)_{|F|}) \quad \text{and} \quad f_{s'} = \rho_{U_{s'}U_s}(f_s) \quad (s, s' \in |F|)\,.$$

It is clear that $\rho_x^\xi = \bigcup_{s\in|F|} \rho_{U_{s'}U_s}(f_s)$. Now let $z \in A$. Then $z \in U_{\rho_x^\xi}$ for any $\xi(F)$ $(F \in \mathfrak{F})$ and, moreover,

$$\rho_x^\xi(z) = \rho_x^\xi \cap \mathcal{S}_z = \rho_{zU_s}(f_s) \quad \text{for} \quad z \in U_s \quad (s \in |F|)\,.$$

Let us show that $\rho_x^\xi(z) = \rho_x^{\xi'}(z)$ for any ξ, ξ'. In fact, let $\xi = (f_s)_{|F|}$, $\xi' = (f'_{s'})_{|F'|}$ and $x = \psi(\xi)$, $x' = \psi(\xi')$. Since $\xi \sim \xi'$, there exists $F^* \in \mathfrak{F}$, where $F^* \succ F'$ and $F^* \succ F'$, such that for each $T^* \in F^*$ we can find $T \in F$ and $T' \in F'$ such that

$$\omega_{TT^*} : T^* \to T\,, \quad \omega_{T'T^*} : T^* \to T' \quad \text{and} \quad \rho_{U_{s^*}U_s}(f_s) = \rho_{U_{s^*}U_{s'}}(f'_{s'})\,,$$

where $s^* \in T^*$. However, $z \in \bigcup_{s^*\in|F^*|} U_{s^*}$, and so it remains to choose $s_0^* \in |F^*|$, such that

$$z \in U_{s_0^*} \quad \text{and} \quad \rho_{zU_s}(f_s) = \rho_{zU_{s'}}(f'_{s'}) \quad (s^* \to s, s^* \to s')\,.$$

Thus $z \in U_{\rho_x}$. Furthermore, let us put $x(z) = \rho_x^\xi(z)|_A$, so that $x(z)$ is a section of $\mathcal{S}$ on A, $x(z) \in \Gamma(A, \mathcal{S})$. In this way we have constructed a morphism $\mathcal{H} : X \to \Gamma(A, \mathcal{S})$. Given $f_A = \mathcal{H}(x)$, $f_A = \mathcal{H}(y)$, let us prove that $x = y$. In fact, at each point $z \in A$ there exists an open ball $B(z, \epsilon)$ of the local homeomorphism $\pi : \mathcal{S} \to \mathcal{D}$ at the point $f_A(z)$. Let us put $U = \bigcup_{z\in A} B(z, \epsilon/2)$ and determine the section $f_z \in \Gamma(B(z, \epsilon/2), \mathcal{S})$ passing through the point $s = f_A(z) \in \mathcal{S}$ such that

$$f_z|_A = f_A|_{B(z,\epsilon/2)}$$

(we note that $\epsilon = \epsilon(z)$). Let

$$B_{ij} = B(z_i, \epsilon_i/2) \cap B(z_j, \epsilon_j/2)\,, \quad B_{ij} \cap A \neq \emptyset\,, \quad z_0 \in B_{ij} \cap A$$

for some $z_i, z_j \in A$. Then $f_{z_i}(z_0) = f_{z_j}(z_0)$, and, consequently, there is an open ball $B_0 \subset B(z_0, \epsilon_0/2)$ of the local homeomorphism at the point $s_0 = f_{z_i}(z_0)$ such that $B_0 \subset B_{ij}$ and $f_{z_i}|_{B_0} = f_{z_j}|_{B_0}$. However, because of the isomorphism $\Gamma(B_{ij}, \mathcal{S}) \to \mathcal{S}_{B_{ij}}$ the holomorphic functions f_{z_i} and f_{z_j} coincide on the connected open set B_{ij} [1, Theorem A6]. The last observation means that $f_{z_i}|_{B_{ij}} = f_{z_j}|_{B_{ij}}$. Now suppose that

$$B_{ij} \cap A = \emptyset\,, \quad \text{but} \quad B'_{ij}(\epsilon_i, \epsilon_j) \cap A \neq \emptyset\,, \quad z' \in B'_{ij} \cap A\,.$$

Clearly, we will obtain by similar reasoning $f'_{z_i}|_{B'_{ij}} = f'_{z_j}|_{B'_{ij}}$. But we have $f'_{z_i}|_{B(z_i,\epsilon_i/2)} = f_{z_i}$ and $f'_{z_j}|_{B(z_j,\epsilon_j/2)} = f_{z_j}$, so that $f_{z_i}|_{B_{ij}} = f_{z_j}|_{B_{ij}}$ (in the case where $B_{ij} \neq \emptyset$). Now there remains the third possibility for $B_{ij} \neq \emptyset$, namely when $B'_{ij} \cap A = \emptyset$. In this case the sections f_{z_i}, f_{z_j} on B_{ij} do not necessarily coincide, therefore let us put $M = \overline{\bigcup B_{ij}}$, where the bar denotes closure in $\mathbb{C}^n$ and the union is taken over all B_{ij} of this third type. It is clear that $M \cap A = \emptyset$, since in the contrary case for $z^* \in M \cap A$ there would exist B^*_{ij} of the third type such that $z^* \in {B_{ij}^*}'$, which is impossible by construction. Let us put $U(f_A) = U \backslash M$, so that $U(f_A) \supset A$ and $U(f_A)$ is an open subset of $\mathbb{C}^n$. Then there exists $f \in \Gamma(U(f_A), \mathcal{S})$ such that $f|_A = f_A$ and, moreover,

$$f|_{U(f_A)\cap B(z,\epsilon/2)} = f_z|_{U(f_A)\cap B(z,\epsilon/2)} \quad (z \in A)\,;$$

also the section on $U(f_A)$ of f with the property $f|_A = f_A$ is uniquely determined (nevertheless, ϕ_A, the corresponding holomorphic function on A, is extended holomorphically to $U(f_A)$ in a manner which, in general, is not unique).

Now if $\psi(\xi) = x$, $\psi(\eta) = y$, it follows from the fact that the family of open sets $\{\bigcup_{s\in|F|} U_s\}_{F\in\mathfrak{F}}$ is fundamental for A that there exists $F^* \in \mathfrak{F}$ such that $U(f_A) \supset \bigcup_{F^*} U_{s^*}$, and moreover by construction

$$\rho_x^\xi|_{\bigcup_{F^*} U_{s^*}} = \rho_y^\eta|_{\bigcup_{F^*} U_{s^*}}\,.$$

The last assertion means that $\xi \sim \eta$ and, consequently, $x = y$. Moreover, the fact that $\{\bigcup_F U_s\}_{F\in\mathfrak{F}}$ is fundamental for A and the constructions carried out above allow us to conclude that $\mathcal{H} : X \to \Gamma(A, \mathcal{S})$ is an isomorphism. The proposition is proved. $\Box$

The absence of sufficiency restrictions on $|\mathfrak{F}|$ in Proposition 1 allows us to apply it to any nonempty $A \subset \mathbb{C}^n$: it is enough to take $\mathfrak{F} = |\mathfrak{F}|$ and U_s $(s \in \mathfrak{F})$ a fundamental system of open sets containing A (in general, of uncountable cardinality). The investigation of topological properties of H-limits in this case produces substantial difficulties, therefore the investigation of $\Gamma(A, \mathcal{S})$ with no more than countable $|\mathfrak{F}|$ is of interest. It is clear that, for example, any closed bounded set $A \subset \mathbb{C}^n$ will be of this type, and the space of sections $\Gamma(A, \mathcal{S})$ is the inductive limit of the sequence of spaces $\Gamma(U_s, \mathcal{S})$ $(s \in |\mathfrak{F}|)$. Furthermore, the corresponding Hausdorff spectrum $\{\Gamma(U_s, \mathcal{S}), \mathfrak{F}, \rho_{U_{s'}U_s}\}$ will be true in this case. In general, the Hausdorff spectrum $\mathcal{X}(\mathcal{S})$ will be true if all open sets U_s $(s \in |\mathfrak{F}|)$ are connected. We recall that each space $\Gamma(U_s, \mathcal{S})$ can be given the separated locally convex topology of uniform convergence on the compact subsets of U_s $(s \in |\mathfrak{F}|)$, under which it is a Fréchet space; we will denote this topology by τ_s.

Proposition 2. *Let $\mathcal{X}(\mathcal{S}) = \{\Gamma(U_s, \mathcal{S}), \mathfrak{F}, \rho_{U_{s'}U_s}\}$ be a true countable Hausdorff spectrum and suppose that $A = \bigcap_{\mathfrak{F}} \bigcup_F U_s$ has a countable fundamental system of compact sets, is connected and $\overset{\circ}{A} \neq \emptyset$. Then the H-limit $X = \underset{\mathfrak{F}}{\overset{\longleftarrow}{\varinjlim}} \rho_{U_{s'}U_s} \Gamma(U_s, \mathcal{S})$ is a separated H-space in the topology τ^* and is continuously embedded in $\mathcal{O}_A$ ($\mathcal{O}_A$ is the algebra of holomorphic functions on A).*

Proof. First of all, by Proposition 1 we have the isomorphism $\mathcal{H} : X \to \Gamma(A, \mathcal{S})$; because of the connectedness of A and the fact that $\overset{\circ}{A} \neq \emptyset$ each holomorphic function on A, $\phi \in \mathcal{O}_A$, is generated by some holomorphic function on the open set $U(\phi)$; moreover, any two holomorphic functions $\phi_1 \in \mathcal{O}_U$ and $\phi_2 \in \mathcal{O}_V$ $(U \supset A,\ V \supset A)$ which coincide on A must coincide on a connected component of the intersection $U \cap V$ (see [1, p. 104]), which also implies the isomorphism $\Gamma(A, \mathcal{S}) \equiv \mathcal{O}_A$. Since A has a countable fundamental system of compact subsets

$$K_n \quad (n = 1, 2, \dots)\,, \quad K_1 \subset K_2 \subset \dots\,,$$

on putting

$$||\phi||_n = \max_{z\in K_n} |\phi(z)| \quad (\phi \in \mathcal{O}_A)\,,$$

we obtain a seminorm on $\mathcal{O}_A$ (or on $\Gamma(A, \mathcal{S})$, which is permissible according to the construction). Furthermore, on putting

$$p(\phi) = \sum_{n=1}^{\infty} 2^{-n} \frac{||\phi||_n}{1 + ||\phi||_n} \quad (\phi \in \mathcal{O}_A)$$

for example, we obtain a quasinorm on $\mathcal{O}_A$ under which $\mathcal{O}_A$ becomes a separated locally convex space with a countable base of neighbourhoods of zero, therefore metrizable, but in general not complete; we will denote this space by $(\mathcal{O}_A, p)$.

We now show that on $\mathcal{O}_A$ the locally convex topology τ^* of the H-limit of the Hausdorff spectrum $\mathcal{X}(\mathcal{S})$ is not weaker than p. In fact, let $W = \{\phi \in \mathcal{O}_A : ||\phi||_N < \epsilon\}$ be some neighbourhood of zero in $(\mathcal{O}_A, p)$ and let $F \in \mathfrak{F}$. Let us choose $s_0 \in |F|$ such that $U_{s_0} \supset K_N$ – this choice turns out to be possible because of the compactness of K_N and the condition $A \subset \bigcup_F U_s$; also we can find a compact set $K^0_m \subset U_{s_0}$ such that $K^0_m \supset K_N$ – here the choice is possible because of the availability of a fundamental system $\{K^0_n\}_{n=1}^{\infty}$ in U_{s_0}. Now it is clear that

$$\mathcal{H} \circ \psi(M^F) \subset W \,,$$

where

$$\xi = (f_s)_F \,,\; M^F = \{\xi \in V_F^{s_0} : \sup_{z \in K^0_m} |f_{s_0}(z)| < \epsilon\} \,,\; \mathcal{H} \circ \psi(\xi) = \phi \,,\; f_{s_0}|_A = \phi \,.$$

Since $\psi(M^F)$ is itself a neighbourhood of zero in the MVG $X_{(F)}$ and $F \in \mathfrak{F}$ was chosen arbitrarily, we have that

$$\mathcal{H}(\mathrm{co} \bigcup_{\mathfrak{F}} \psi(M^F)) \subset W$$

and is a neighbourhood of zero in the topology τ^*. This also shows that $\tau^* \geq p$. The proposition is proved. □

The conditions of Proposition 2 are satisfied, for example, by $A = \overline{\Delta}(0, r)$, the compact polydisk in $\mathbb{C}^n$, or by any domain $\mathcal{D} \subset \mathbb{C}^n$. It is not difficult to see that if A is a connected set and $A = \bigcap_{\mathfrak{F}} \bigcup_F U_s$, where $\mathfrak{F}$ is an admissible class for Ω, then without loss of generality we may assume that the U_s $(s \in |\mathfrak{F}|)$ are connected open sets in $\mathbb{C}^n$. In fact, let each U_s have nonempty intersection with A, which is natural and can always be arranged by the method of transformation of indices $(s \in |\mathfrak{F}|)$. Let us denote by $\widetilde{U}_s$ the open connected component of U_s which contains $U_s \cap A$ $(s \in |\mathfrak{F}|)$. Now it is clear that $A = \bigcap_{\mathfrak{F}} \bigcup_F \widetilde{U}_s$ for the admissible class $\mathfrak{F}$ in Ω and moreover, if $\{\bigcup_F U_s\}_{\mathfrak{F}}$ were a fundamental system of neighbourhoods for A, then $\{\bigcup_F \widetilde{U}_s\}_{\mathfrak{F}}$ would be the same. Now let us consider the question: For what classes of sets $A \subset \mathbb{C}^n$ do we have a representation

$$A = \bigcap_{F \in \mathfrak{F}} \bigcup_{s \in F} U_s \,,$$

where $\mathfrak{F}$ is an admissible class for Ω (a countable set)?

Let A be any nonempty bounded connected subset of $\mathbb{C}^n$ and $B = B(z_0, r)$ an open ball such that $\overline{A} \subset B$. By Proposition 3.2 for the s-set $B \backslash A$ we have the representation

$$B \backslash A = \bigcup_{\widehat{B} \in \mathcal{K}} \bigcap_{t \in \widehat{B}} L_t \,,$$

where the L_t are open subsets of $B(z_0, r)$ ($t \in |\mathcal{K}|$, $|\mathcal{K}|$ a countable set) and $\mathcal{K}$ is some family of subsets $\widehat{B} \subset \Omega$; moreover, for each $\widehat{B} \in \mathcal{K}$ the intersections $\{\bigcap_{\widehat{B}} L_t\}_{\mathcal{K}}$ form a fundamental system of compact subsets of $B \backslash A$. Since $\mathbb{C}^n$ is a finite-dimensional space, by Proposition 3.10 we will obtain the representation

$$B\backslash A = \bigcup_{\widehat{B}\in\mathcal{K}} \bigcap_{t\in\widehat{B}} \overline{L}_t \,,$$

where the $\overline{L}_t \subset B(z_0, r+\epsilon)$ are compact sets ($t \in |\mathcal{K}|$). Now

$$A = B\backslash \bigcup_{\mathcal{K}} \bigcap_{\widehat{B}} \overline{L}_t = \bigcap_{\mathcal{K}} (B\backslash \bigcap_{\widehat{B}} \overline{L}_t) = \bigcap_{\mathcal{K}} \bigcup_{\widehat{B}} (B\backslash \overline{L}_t) \,,$$

and $G_t = B\backslash\overline{L}_t$ is an open set in $\mathbb{C}^n$ ($t \in |\mathcal{K}|$). We will show that $\{\bigcup_{\widehat{B}} \widetilde{G}_t\}_{\mathcal{K}}$ form a fundamental system of open connected neighbourhoods of A. In fact, if $W \supset A$, $W \subset B$ is an open set, then without loss of generality we may assume that $W \subset B(z_0, r-\delta)$ for some $\delta > 0$, so that $P = (B(z_0, r-\delta) \cup \partial B)\backslash W$ is a compact subset of $B(z_0, r)$. Therefore there exists a compact subset $\bigcap_{\widehat{B}_0} \overline{L}_t = \bigcap_{\widehat{B}_0} L_t$ such that $P \subset \bigcap_{\widehat{B}_0} \overline{L}_t$ and, consequently,

$$B\backslash P \supset \bigcup_{\widehat{B}_0} (B\backslash \overline{L}_t) \quad \text{or} \quad W \cup \{z : r-\delta < |z-z_0| < r\} \supset \bigcup_{\widehat{B}_0} G_t \supset \bigcup_{\widehat{B}_0} \widetilde{G}_t \,.$$

However, because of the connectedness of $\widetilde{G}_t$ and the ordering of $\widehat{B}_0 \in \mathcal{K}$ we obtain the inclusion $W \supset \bigcup_{\widehat{B}_0} \widetilde{G}_t$, which was to be established.

Thus we obtain the following

Proposition 3. *Every connected bounded subset $A \subset \mathbb{C}^n$ has a representation*

$$A = \bigcap_{F\in\mathfrak{F}} \bigcup_{s\in F} U_s \,, \tag{6}$$

where $\mathfrak{F}$ is an admissible class for the countable set Ω and the U_s are connected open subsets (domains) in $\mathbb{C}^n$.

In particular, for such a set A the Hausdorff spectrum

$$\mathcal{X}(\mathcal{S}) = \{\Gamma(U_s, \mathcal{S}), \mathfrak{F}, \rho_{U_{s'}U_s}\}$$

is true (it suffices to apply the uniqueness theorem for holomorphic functions). In the representation (6) it is natural to require that if $U_s \cap U_{s'} \neq \emptyset$ ($s, s' \in |\mathfrak{F}|$) then it is a connected set. Only such sets A will be considered further.

In what follows the space $\mathcal{O}_A$ of germs of holomorphic functions on A will be provided with the topology p (in general not separated) of uniform convergence on the compact subsets of A and with the locally convex topology of the H-limit. As has already been noted above (Proposition 1), for a connected bounded subset $A \subset \mathbb{C}^n$ we have the linear isomorphism

$$X \equiv \Gamma(A, \mathcal{S}) \equiv \mathcal{O}_A \,.$$

We also note that if the set A has an interior point then $\mathcal{O}_A$ coincides with the space of holomorphic functions on A (up to isomorphism).

Weierstrass's Global Division Theorem

Weierstrass's preparation theorem and the division theorem for germs of holomorphic functions at a point $w \in \mathbb{C}^n$ allow us to establish a series of properties of local rings ${}_n\mathcal{O}_w$ and modules over these rings (Noetherian, Oka's Lemma on the exactness of homomorphisms of ${}_n\mathcal{O}$-modules, etc. [1]). The proofs have a number of algebraic characteristics, therefore consideration of a global variant of the theorems is significantly different and uses topological results of linear analysis (see, for example, [1]). A more careful analysis makes it possible to formulate a global division theorem in terms of the existence and continuity of a linear operator acting on locally convex spaces so that the local and global variants of Weierstrass's theorem turn out to be in fact special cases of a more general theorem. In this section we obtain a stronger form of Theorems II.B.3 and II.D.1 in [1] for the case of H-spaces. $\mathbb{C}^{n-1}_m$ denotes $\underbrace{\mathbb{C} \times \cdots \times \mathbb{C}}_{m-1} \times \underbrace{\mathbb{C} \times \cdots \times \mathbb{C}}_{n-m}$ and $\pi_m : \mathbb{C}^n \to \mathbb{C}^{n-1}_m$ is the projection of $\mathbb{C}^n$ onto $\mathbb{C}^{n-1}_m$; at the same time $\pi^m : \mathbb{C}^n \to \mathbb{C}_m$ and $\mathbb{C}^n = \mathbb{C}^{n-1}_m \times \mathbb{C}_m$ so that π^m is the projection of $\mathbb{C}^n$ onto $\mathbb{C}_m$. For notational convenience in what follows the germ of a holomorphic function is denoted by capital Roman letters F, G, H, $\ldots$.

We will say that the germ $H \in \mathcal{O}_A$ ($A \subset \mathbb{C}^n$) is a *w-local Weierstrass polynomial in z_m* ($1 \le m \le n$) *of degree k* ($k > 0$) if there exists $w \in A$ and a function $h \in H$ which is holomorphic on an open neighbourhood $U \supset A$ and has representation on U

$$\begin{aligned} h(z) &= (z_m - w_m)^k + a_1(z')(z_m - w_m)^{k-1} + \cdots + a_k(z')\,, \\ z' &= (z_1, z_2, \ldots, z_{m-1}, z_{m+1}, \ldots, z_n)\,, \end{aligned} \tag{7}$$

where the $a_j(z')$ are holomorphic functions on $\pi_m(U)$, $a_j(w') = 0$, and $w = w' \times w_m$ ($j = 1, 2, \ldots, k$). It is clear that the holomorphic function h is regular of order k in z_m at the point $w \in A$.

Theorem 1. (Weierstrass's global division theorem.) *Let $A \subset \mathbb{C}^n$ be a nonempty connected bounded set such that $\pi^m(A)$ is closed and let $H \in \mathcal{O}_A$ be a w-local Weierstrass polynomial in z_m of degree k ($k > 0$) with representation $h_U \in H$ such that*

$$\{z \in \pi_m^{-1} \circ \pi_m(A) \cap U : h_U(z) = 0\} \subset A\,.$$

Then there exists a continuous linear operator $L : \mathcal{O}_A \to \mathcal{O}_A \times \mathcal{O}_A$, where

$$L(F) = (G, P)\,, \qquad F = GH + P\,,$$

$$P = \sum_{j=0}^{k-1} P_j(z') z_m^j\,, \qquad P_j \in \mathcal{O}_A\,.$$

First of all we recall [1, Chapter 2, §5] that $\mathcal{O}_A$ has the topology p of uniform convergence on the compact subsets, which in general is neither separated nor complete, and $\mathcal{O}_A \times \mathcal{O}_A$ has the usual product topology. In the course of the proof of Theorem 1 $\mathcal{O}_A$ will also be given another stronger locally convex topology, again in general not separated, under which it is an H-space. Therefore we first present a lemma for Theorem 1.

Lemma 1. *Let $A : X \to Y$ be a closed linear operator, where X is an H-space under the locally convex topology τ^* and (Y, σ) is an H-space (in general X, Y are not separated spaces). Then A is continuous.*

Proof. Let M, N be the respective nonseparated parts of X, Y and X/M, Y/N the separated quotient spaces with quotient maps $\xi : X \to X/M$ and $\eta : Y \to Y/N$. Then the quotient topology $\xi\tau^*$ on X/M is in general weaker than the topology $(\xi\tau^*)^*$, the limit of the corresponding Hausdorff spectrum (see, for example, [4]); let $\eta\sigma$ be the quotient topology on Y/N. Then the diagram

$$\begin{array}{ccc} X/M & \overset{A^*}{\cdots\cdots\cdots>} & Y/N \\ \xi\Big\uparrow & & \Big\uparrow\eta \\ X & \xrightarrow{\quad A \quad} & Y \end{array} \tag{8}$$

is commutative and the induced mapping A^* exists because of the closedness of the operator A. In fact, the closedness of A implies that $N = \bigcap\limits_{U\in\mathcal{U},V\in\mathcal{V}} \{U + AV\}$, where $\mathcal{U}$, $A\mathcal{V}$ are bases of neighbourhoods of zero for the topologies σ, $A\tau^*$ respectively. But $AM \subset AV$ for any $V \in \mathcal{V}$ and $0 \in U$, therefore $AM \subset U + AV$ $(\forall\ U, V)$ and, consequently, $AM \subset N$. Moreover, the induced mapping A^* is clearly linear; we will show that A^* is a closed operator. For this we have to show that

$$0 = \bigcap_{U\in\mathcal{U},V\in\mathcal{V}} \{\eta U + A^*\xi V\}\,.$$

Since $\eta A = A^*\xi$, this is equivalent to the relation $0 = \bigcap_{\mathcal{U},\mathcal{V}} \eta\{U + AV\}$; let us suppose that $a \in \bigcap_{\mathcal{U},\mathcal{V}} \eta\{U + AV\}$. Then $\eta^{-1}a \cap (U + AV) \neq \emptyset$ $(\forall\ U, V)$. But $\eta^{-1}a = y + N$ and because of the absolute convexity of $U + AV$ and Theorem 1.3 of [2] we obtain $\eta^{-1}a \subset U + AV$ $(\forall\ U, V)$. This implies that $\eta^{-1}a \subset N$; consequently $a = 0$ and A^* is a closed operator.

Thus by the Closed Graph Theorem for the H-space $(Y/N, \eta\sigma)$ and complete MVGs the closed operator A^* is continuous from $(X/M, (\xi\tau^*)^*)$ to $(Y/N, \eta\sigma)$. The existence of the Hausdorff spectrum for $(Y/N, \eta\sigma)$ follows from Proposition 4.10 and [4].

Now we will establish the continuity of the operator $A : X \to Y$. Let W be a closed absolutely convex neighbourhood of zero in Y and (V_n^F) a base of absolutely convex neighbourhoods of zero in the TVG $X_{(F)}$ $(F \in \mathfrak{F})$, where

$$X = \bigcup_{F\in\mathfrak{F}} \bigcap_{s\in F} X_s\,.$$

If it is shown that $A : X_{(F)} \to Y$ is continuous, then by the definition of the topology τ^* and the local convexity of (Y, σ) this will imply that $A : X \to Y$ is continuous. Therefore let $F \in \mathfrak{F}$ be fixed. Then (ξV_n^F) is a base of neighbourhoods of zero for the TVG $(X/M)_{(F)}$ (see Proposition 4.10) and ηW is a neighbourhood of zero in $(Y/N, \eta\sigma)$. By the commutativity of Diagram (8) $A^*\xi V_n^F = \eta A V_n^F$ $(\forall\, n \in \mathbb{N})$ and by the continuity of A^* there exists $\overline{N} \in \mathbb{N}$ such that $A^*\xi V_{\overline{N}}^F \subset \eta W$ or $\eta A V_{\overline{N}}^F \subset \eta W$. Hence, $A V_{\overline{N}}^F \subset W + N$, but since W is a closed set and $N \subset W$, then $W + N \subset W$ and the continuity of $A : X_{(F)} \to Y$ is established. This means that $A : X \to Y$ is continuous and the lemma is proved.

Lemma 2. *Let $L : (\mathcal{O}_A, p) \to (\mathcal{O}_A, p)$ be a closed linear operator. Then $L : (\mathcal{O}_A, p^*) \to (\mathcal{O}_A, p^*)$ is continuous (A is a nonempty connected bounded subset of $\mathbb{C}^n$).*

Proof. We recall that the locally convex topology p^* of the H-limit of a Hausdorff spectrum on the space of germs of holomorphic functions on A is not weaker than the locally convex topology p of uniform convergence on the compact subsets of A. Therefore the operator $L : (\mathcal{O}_A, p^*) \to (\mathcal{O}_A, p^*)$ is closed. Moreover, by Proposition 3.10 the set A has a representation

$$A = \bigcap_{F\in\mathfrak{F}} \bigcup_{s\in F} U_s\,,$$

where $\mathfrak{F}$ is an admissible class for the countable set Ω and U_s is a domain in $\mathbb{C}^n$; moreover, each U_s ($s \in |\mathfrak{F}|$) has a countable fundamental system of compact subsets $(K_n^s)_{n=1}^\infty$ with $K_1^s \subset K_2^s \subset \dots$. We will show that each space $(\mathcal{O}_A)_{(F)}$ ($F \in \mathfrak{F}$) is complete and so $(\mathcal{O}_A, p^*)$ is an H-space.

We recall that

$$X = \bigcup_{F\in\mathfrak{F}} \bigcap_{s\in F} \psi(V_F^s)$$

and $\kappa : X \to \Gamma(A, \mathcal{S}) \equiv \mathcal{O}_A$ is an isomorphism. The TVG $(\mathcal{O}_A)_{(F)}$ is an isomorphic image of the restriction of the complete TVG of countable character $S_{(F)}$ (notation of 3.2) to X. Therefore it is enough to establish the closedness of $\kappa^{-1}(\mathcal{O}_A)_{(F)}$ in $S_{(F)}$. The arguments are carried out more easily for the germs of holomorphic functions on A.

Let $F \in \mathfrak{F}$, $U_F \supset A$, $U_F = \bigcup_{s\in F} U_s$ (F is no more than countable and is totally linearly ordered for s). Further, let (G_n) be a sequence of germs of holomorphic functions on A which is fundamental in $(\mathcal{O}_A)_{(F)}$. Since $(\mathcal{O}_A)_{(F)}$ is a quotient group (up to isomorphism) of the complete MVG $(\prod_F \mathcal{O}_{U_s})_{(F)}$, where $\mathcal{O}_{U_s}$ is the Fréchet space with the topology of uniform convergence on the compact sets $(K_n^s)_1^\infty$, it follows from Proposition 4.10 that there exists a subsequence $g_{n_k} \in \prod_F \mathcal{O}_{U_s}$ ($k = 1, 2, \dots$) such that g_{n_k} converges in $(\prod_F \mathcal{O}_{U_s})_{(F)}$ to some element $g \in \prod_F \mathcal{O}_{U_s}$ and $\psi g_{n_k} = G_{n_k}$ ($k = 1, 2, \dots$). The last condition implies in particular that $g_{n_k} = (f_s^{n_k})_{s\in F}$, where $f_p^{n_k}|_{U_s} = f_s^{n_k}$ ($s \le p$, $p \in F$), $p = p(k)$, ($k = 1, 2, \dots$). Put $p_0 = \inf_k p(k)$, $p_0 \in F$. Then, clearly, $f_{p_0}^{n_k}|_{U_s} = f_s^{n_k}$ ($s \le p_0$, $k = 1, 2, \dots$); we will denote by $f_k = f_{p_0}^{n_k}$ the holomorphic functions on the open connected set U_{p_0} ($k = 1, 2, \dots$). Since $\lim_{k\to\infty} g_{n_k} = g$ and $g = (\hat{g}_s)_{s\in F}$, then, in particular, f_k converges to $\hat{g}_{p_0}$ in $\mathcal{O}_{U_{p_0}}$ and moreover $g - g_{n_{k_l}} \in V_F^s$ ($s \in F$, $n_{k_l} = n_{k_l}(s)$, $l = 1, 2, \dots$). The last observation means that for $s > p_0$ the holomorphic function $\hat{g}_{p_0} - f_{n_{k_l}}$ has a unique extension to the set U_s ($s \in F$). However, each element g_{n_k} is equivalent to elements $a_k \in \prod_{F_k} \mathcal{O}_{U_s}$, i.e. $\psi g_{n_k} = \psi a_k$ and moreover $a_k \in \bigcap_{s\in F_k} V_{F_k}^s$ ($k = 1, 2, \dots$). Furthermore, we may assume without loss of generality that $F_1 \prec F_2 \prec \dots$. Thus the holomorphic function $f_{n_{k_l}}$ has a unique extension to the set $U_{F_{k_l}} \supset A$ ($l = 1, 2, \dots$) and, consequently, the holomorphic function $\hat{g}_{p_0}$ has a unique extension to the set $U_s \cap U_{F_{k_l}}$ ($s > p_0, s \in F$). Since $U_s \cap U_{F_{k_l}} = \bigcup_{q\in F_{k_l}} (U_s \cap U_q)$ and $U_s \cap U_q \subset U_s \cap U_{q'}$ ($q \le q'$), then $U_s \cap U_{F_{k_l}}$ is a connected open set ($l = 1, 2, \dots, k_l = k_l(s)$), and since the set $\{s \in F : s > p_0\}$ can be enumerated, let its points be $s_1, s_2, \dots$.

Thus on each nonempty open connected set $U_s \cap U_{F_{k_l}}$ a holomorphic function $\hat{g}_{p_0 s}$ is defined such that $\hat{g}_{p_0 s}|_{U_{p_0}} = \hat{g}_{p_0}$ ($s > p_0, s \in F$). But since each nonempty intersection $(U_{s_i} \cap U_{F_{k_i}}) \cap (U_{s_j} \cap U_{F_{k_j}})$ is connected by construction and has nonempty intersection with U_{p_0}, then a holomorphic function $\hat{g}$ is defined on the open set $\bigcup_{i=1}^\infty (U_{s_i} \cap U_{F_{k_i}})$ such that $\hat{g}|_{U_{s_i} \cap U_{F_{k_i}}} = \hat{g}_{p_0 s_i}$ ($i = 1, 2, \dots$). Then $\hat{g}|_{U_{F^*}}$ generates an element of $\bigcap_{s\in F^*} V_{F^*}^s$ such that $\psi g = \psi \hat{g}|_{U_F}$ and, consequently, $\psi g = G \in \mathcal{O}_A$ and $\lim_{n\to\infty} G_n = G$ in the TVG $(\mathcal{O}_A)_{(F)}$. Thus the space $(\mathcal{O}_A)_{(F)}$ is complete and $(\mathcal{O}_A, p^*)$ is an H-space ($U_{F^*} \subset \bigcup_{i=1}^\infty (U_{s_i} \cap U_{F_{k_i}})$, $F^* \in \mathfrak{F}$).

Continuity of the operator A now follows from the Closed Graph Theorem, Lemma 1 and the closedness of the operator $A : (\mathcal{O}_A, p^*) \to (\mathcal{O}_A, p^*)$. The lemma is proved.

Proof of Theorem 1. Let $H \in \mathcal{O}_A$ be a w-local Weierstrass polynomial in z_m of degree k and let $h \in H$ be a holomorphic function on the open connected set $U \supset A$ which satisfies the conditions of the theorem and the relation (7). Furthermore, let $F \in \mathcal{O}_A$ be an arbitrary germ, let $f \in F$ and suppose that f is a holomorphic function on the domain $V \subset U_1$ (it may be assumed without loss of generality that $\overline{U}_1 \subset U$). Let us fix a point $a' \in \pi_m(A)$ and a closed (according to the condition) cross-section $r_{a'}(A) \subset r_{a'}(U)$ and choose a closed piecewise-smooth Jordan contour $\Gamma_{a'}$ which encloses $r_{a'}(A)$ and lies in $r_{a'}(V)$ and has length $l(\Gamma_{a'})$. Since the function $h(z)$ is continuous on the open neighbourhood of the compact set

$$Q = \{z \in V : z_m \in \Gamma_{a'}\,,\ \pi_m(z) = a'\}\,,$$

there exists an open ball $B(0, \delta)$ such that for $z_m \in \Gamma_{a'}$ and $z' \in \pi_m[(a', z_m) + B(0, \delta)]$ we have the inequality

$$|h(z) - h(a', z_m)| \le \inf_{r_{a'}} |h(a', z_m)| \tag{9}$$

and the inclusion

$$(a', z_m) + B(0, \delta) \subset V \quad (z_m \in \Gamma_{a'})\,.$$

Moreover, by the compactness of Q we can choose a polydisk $\triangle'(0, \delta'_{a'}) \subset \mathbb{C}^{n-1}$ such that

$$[a' + \triangle'(0, \delta'_{a'})] \times \Gamma_{a'} \subset \bigcup_{z_m \in \Gamma_{a'}} [(a', z_m) + B(0, \delta)]\,. \tag{10}$$

In fact, we cover the compact set Q with the open balls $(a', z_m) + B(0, \delta)$ $(z_m \in \Gamma_{a'})$, in each of which we choose a polydisk $(a', z_m) + \triangle(0, \delta_{a'})$ $(z_m \in \Gamma_{a'},\ \delta_{a'} = (\delta'_{a'}, \delta_m))$ with these taken together also covering Q. Put $R_{a'} = a' + \triangle'(0, \delta'_{a'})$. Then

$$\bigcup_{z_m \in \Gamma_{a'}} [(a', z_m) + \triangle(0, \delta_{a'})] = \Big[\bigcup_{r_{a'}} (z_m + \triangle(0, \delta_m))\Big] \times R_{a'} \supset \Gamma_{a'} \times R_{a'}\,,$$

from which (10) follows.

The inclusion (10) allows us to conclude in particular that (9) and the inclusion $\Gamma_{a'} \subset r_{z'}(V)$ are valid for $z' \in R_{a'}$. Now for the indicated $z' \in R_{a'}$ the function $h_{z'} = h(z', z_m)$ as a holomorphic function of one variable z_m has exactly k zeros inside the contour $\Gamma_{a'}$ by Rouché's Theorem for the domain $r_{z'}(V) \cap r_{a'}(V)$; in particular, $h_{z'} \ne 0$ on $\Gamma_{a'}$ and outside this contour in the domain $r_{z'}(V)$ (and even $r_{z'}(U)$).

We will denote by $\mathcal{D}_{a'}$ the domain bounded by $\Gamma_{a'}$ and put

$$\mathcal{D} = \bigcup_{a' \in \pi_m(A)} \left(\mathcal{D}_{a'} \times R_{a'}\right).$$

It is clear that $\mathcal{D}$ is an open connected set such that $A \subset \mathcal{D} \subset V \subset U_1$.

Further, for each open set $\mathcal{D}_{a'} \times R_{a'}$ $(a' \in \pi_m(A))$ we define a holomorphic function (see [1])

$$g_{a'}(z) = \frac{1}{2\pi i} \int_{\Gamma_{a'}} \frac{f(z', \zeta)}{h(z', \zeta)} \cdot \frac{d\zeta}{\zeta - z_m}$$

and a holomorphic function $p_{a'}(z) = f(z) - g_{a'}(z)h(z)$. Therefore

$$p_{a'}(z) = \frac{1}{2\pi i}\int_{\Gamma_{a'}} \frac{f(z',\zeta)}{h(z',\zeta)}\left[\frac{h(z',\zeta)-h(z',z_m)}{\zeta - z_m}\right] d\zeta$$

where

$$p_{a'}(z) = \sum_{j=0}^{k-1} p_{a'_j}(z')(z_m - w_m)^j\,,$$

$$p_{a'_j} = \frac{1}{2\pi i}\int_{\Gamma_{a'}} \frac{h_j^*(z',\zeta)}{h(z',\zeta)}\, f(z',\zeta)\, d\zeta \quad (j = 0, 1, \ldots, k-1)\,,$$

and the holomorphic functions h_j^* $(j = 0, 1, \ldots, k-1)$ are defined from (7) by consideration of the expression

$$\frac{h(z',\zeta)-h(z',z_m)}{\zeta - z_m}\,.$$

The uniqueness of the functions $g_{a'}$ and $p_{a'}$ is established similarly to [1, p. 93] by using Rouché's Theorem.

If $\big(\mathcal{D}_{a'} \times R_{a'}\big) \cap \big(\mathcal{D}_{a''} \times R_{a''}\big) \neq \emptyset$, then for $\hat{z} \in \big(\mathcal{D}_{a'} \times R_{a'}\big) \cap \big(\mathcal{D}_{a''} \times R_{a''}\big)$ we have $\hat{z}_m \in \mathcal{D}_{a'} \cap \mathcal{D}_{a''}$. Because the contours $\Gamma_{a'}$ and $\Gamma_{a''}$ are homotopic this implies that the following identity holds:

$$\int_{\Gamma_{a'}} \frac{f(\hat{z}',\zeta)}{h(\hat{z}',\zeta)} \cdot \frac{d\zeta}{\zeta - \hat{z}_m} = \int_{\Gamma_{a''}} \frac{f(\hat{z}',\zeta)}{h(\hat{z}',\zeta)} \cdot \frac{d\zeta}{\zeta - \hat{z}_m}\,.$$

Thus $g_{a'}(\hat{z}) = g_{a''}(\hat{z})$ and, consequently, a holomorphic function $g(z)$ can be defined on the domain $\mathcal{D}$ such that $g|_{R_{a'}\times\mathcal{D}_{a'}} = g_{a'}$ $(a' \in \pi_m(A))$. In the same way a holomorphic function $p(z)$ can be defined such that $p|_{R_{a'}\times\mathcal{D}_{a'}} = p_{a'}$ $(a' \in \pi_m(A))$ and

$$p(z) = \sum_{j=0}^{k-1} p_j(z')(z_m - w_m)^j\,,$$

so that we have the unique representation

$$f(z) = g(z)h(z) + p(z) \quad (z \in \mathcal{D})\,. \tag{11}$$

Thus a linear operator $L : \mathcal{O}_A \to \mathcal{O}_A \times \mathcal{O}_A$ is defined by the relation $L(F) = (G, P)$, $F = GH + P$, $f \in F$, $g \in G$, $h \in H$, $p \in P$. The operator L has components $L_1 : F \to G$ and $L_2 : F \to P$, whose continuity in the respective topologies will also imply that of L. Let us therefore investigate the continuity of the operators L_1 and L_2.

It follows clearly from the relation (11) that L_1 and L_2 are closed linear operators from $(\mathcal{O}_A, p)$ into $(\mathcal{O}_A, p)$. Thus by Lemma 2 the operator $L_i : (\mathcal{O}_A, p^*) \to (\mathcal{O}_A, p^*)$ is continuous $(i = 1, 2)$, as also is the operator

$$L : (\mathcal{O}_A, p^*) \to (\mathcal{O}_A, p^*) \times (\mathcal{O}_A, p^*)\,.$$

We now establish the continuity of the operator $L : (\mathcal{O}_A, p) \to (\mathcal{O}_A, p) \times (\mathcal{O}_A, p)$. First of all, let us fix an open set $\mathcal{D}^h$ constructed be the method indicated above for the holomorphic function $h(z)$ on the domain U_1; then by the compactness of A we choose a finite subcover $\bigcup_{i=1}^{N}\left(R^h_{a'_i}\times\mathcal{D}^h_{a'_i}\right)$, where by the construction it may be assumed without loss of generality that on the distinguished boundary of the polydisk $R^h_{a'_i}$ the function $h(z',\zeta)=0$ only for $\zeta\in\mathcal{D}^h_{a'_i}$ $(i=1,2,\dots,N)$. Therefore $h(z)\neq 0$ on the distinguished boundary of the polydomain $R^h_{a'_i}\times\mathcal{D}^h_{a_i}$, which is part of the boundary of the domain $\bigcup_{i=1}^{N}\left(R^h_{a'_i}\times\mathcal{D}^h_{a'_i}\right)=U_2$. If we now put

$$M=\sup_{0\le j\le k-1}\ \sup_{\bigcup_{i=1}^{N}\left(\overline{R^h_{a'_i}}\times\Gamma_{a'_i}\right)}\left|\frac{h^*_j(z',\zeta)}{h(z',\zeta)}\right|,$$

then $M<+\infty$.

Now let $F\in\mathcal{O}_A$, $L_1(F)=G$, $L_2(F)=P$ and choose $f\in F$ with domain of definition $V\subset U_2$; construct the domain $\mathcal{D}^f\subset V$ such that $\overline{\mathcal{D}^f}\subset V$, while the functions $p(z)$ and $g(z)$ are defined on $\mathcal{D}^f$ $(p\in P,\ g\in G)$ and the relation (11) holds. It is clear that $\mathcal{D}^f\supset A$ and the following diagrams are commutative:

$$\begin{array}{ccc} F & \overset{L_2}{\cdots\cdots>} & P \\ \Big\uparrow & & \Big\uparrow \\ f & \longrightarrow & p \end{array}\quad,\qquad \begin{array}{ccc} F & \overset{L_1}{\cdots\cdots>} & G \\ \Big\uparrow & & \Big\uparrow \\ f & \longrightarrow & g \end{array}\ .$$

We will establish the continuity of the operator $L_2:(\mathcal{O}_A,p)\to(\mathcal{O}_A,p)$, the continuity of L_1 being obvious. Let $a'\in\pi_m(A)$. Then

$$\begin{aligned}|p_{a'}(z)| &\le K\sum_{j=0}^{k-1}|p_{a'_j}(z')|\le\frac{K\cdot M}{2\pi}\sum_{j=0}^{k-1}\int_0^{l(\Gamma^f_{a'})}|f(z',\zeta)|\cdot|d\zeta|\\ &\le \frac{K\cdot M}{2\pi}\cdot k\cdot\sup_{\overline{\mathcal{D}^f}}|f(z',\zeta)|\cdot l(\Gamma^f_{a'})\,.\end{aligned}$$

Now we choose a sequence $V_1=V\supset V_2\supset\dots$ which is fundamental for A and compact sets $\overline{\mathcal{D}}_m=\overline{\mathcal{D}^f_m}$, where $\overline{\mathcal{D}^f_m}\subset V_m$ such that $\overline{A}=\bigcap_{m=1}^{\infty}\overline{\mathcal{D}^f_m}$ and, moreover, the sequence $(\overline{\mathcal{D}}_m)$ converges to $\overline{A}$ in the Hausdorff metric for all compact subsets of $\mathbb{C}^n$. This means in particular that for $f\in\mathcal{O}_{V_1}$ we have the relation

$$\overline{\lim_{m\to\infty}}\sup_{\overline{\mathcal{D}}_m}|f(z)|\le\sup_A|f(z)|\,.$$

In fact, let us assume the contrary, i.e. there exist $\epsilon>0$ and a sequence (m_k) such that

$$\sup_A|f(z)|+\epsilon<\sup_{\overline{\mathcal{D}}_{m_k}}|f(z)|\quad(k\in\mathbb{N})\,.$$

From this we find a sequence (z_{m_k}) such that $z_{m_k}\in\overline{\mathcal{D}}_{m_k}$ and

$$\sup_A|f(z)|+\epsilon<|f(z_{m_k})|\quad(k\in\mathbb{N})\,;$$

but then we can find a subsequence $(z_{m_{k_l}})$ such that $z^*=\lim\limits_{l\to\infty}z_{m_{k_l}}$. Then $z^*\in\overline{A}$ and, consequently, we have the inequality

$$\sup_A|f(z)|+\epsilon\le|f(z^*)|\,,$$

which is impossible.

Therefore

$$\begin{aligned} ||P||_A &= \sup_A |p(z)| \\ &\leq \overline{\lim}_{m\to\infty} \sup_{\mathcal{D}_m} |p(z)| = \overline{\lim}_{m\to\infty} \sup_{\mathcal{D}_m} |p_{a'}(z)| \\ &\leq \tfrac{K\cdot M\cdot k}{2\pi}\, \overline{\lim}_{m\to\infty} l(\Gamma^m_{a'}) \cdot \overline{\lim}_{m\to\infty} \sup_{\overline{\mathcal{D}}_m} |f(z)| \\ &\leq K_A \cdot l_A \cdot \sup_A |f(z)| = K_A \cdot l_A \cdot ||F||_A\,. \end{aligned}$$

Thus the operator $L_2 : (\mathcal{O}_A, p) \to (\mathcal{O}_A, p)$ is continuous. The theorem is proved. □

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Ю.В. Бондаренко

СИЛЬНОЕ УСЛОВИЕ ШОКЕ ДЛЯ КОНУСОВ В ПРОСТРАНСТВЕ ФУНКЦИЙ

В настоящей статье приведены некоторые теоремы о представлении конусов в пространстве функций на (0;?). Эти конструкции навеяны, с одной стороны , классической теоремой Каратеодори-Минковского о представлении элементов конуса через крайние точки , а с другой стороны, - конструкциями из работ, посвященных операторному представлению конусов убывающих и вогнутых функций в весовом пространстве.

Ключевые слова: конус в пространстве функций, крайние лучи, весовые пространства, конуса убывающих и вогнутых функций.

Ju.V.Bondarenko

STRONG CONDITION SHOKE FOR CONES IN SPACE OF FUNCTIONS

Some theorems about representation of cones in function spaces on (0;?) are considered. We use the classical Karatheodory – Minkowski theorem about representation of cone elements by extremal points and operator representation of cones of monotone and concave functions in weight spaces.

Key words: cones in function spaces, extremal points, weight spaces, cones of monotone and concave functions.