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The Category of Hausdorff Spectra over a Semiabelian Category

In this article the category $\mathcal{H}$ of Hausdorff spectra is introduced into the discussion by means of an appropriate factorization of the category of Hausdorff spectra $\mathrm{Spect}\,\mathcal{G}$ over the category $\mathcal{G}$. If $\mathcal{G}$ is a semiabelian complete subcategory of the category TG, then $\mathcal{H}$ is a semiabelian category (in the sense of V. P. Palamodov [1]).

Key words: Hausdorff spectra, semiabelian category, commutative diagram.

Let $\mathcal{X} = \{X_s, \mathfrak{F}, h_{s's}\}$ and $\mathcal{Y} = \{Y_p, \mathfrak{F}^1, h_{p'p}\}$ be Hausdorff spectra over some category $\mathcal{G}$. We will call any set of morphisms $\omega_{ps} : X_s \to Y_p$ of the category $\mathcal{G}$ which satisfies the following conditions a *mapping of spectra* $\omega_{\mathcal{Y}\mathcal{X}} : \mathcal{X} \to \mathcal{Y}$:

(1) there exist mappings $\varphi : \mathfrak{F} \to \mathfrak{F}^1$, $\phi : F^1 \to F$ $(\forall\, F \in \mathfrak{F})$, $\chi^{T^1} : T^1 \to T$ $(\forall\, T^1 \in F^1)$, $\chi^{T^1}|_{T^1_0} = \chi^{T^1_0}$, $T^1_0 \subset T^1$ such that ($\mapsto$ denotes mapping of elements)

$$\begin{tikzcd}
s & \in & T & \in & F & \in & \mathfrak{F} \arrow[d, "\varphi"] \\
p \arrow[u, "\chi"] & \in & T^1 \arrow[u, shift left, "\chi"] \arrow[u, shift right, "\Phi"'] & \in & F^1 \arrow[u, "\Phi"] & \in & \mathfrak{F}^1
\end{tikzcd}$$

with also $F \xrightarrow{\varphi} F^1$.

(2) for each pair $(p, \chi(p))$ a morphism $\omega_{p\chi(p)} : X_{\chi(p)} \to Y_p$ of the category $\mathcal{G}$ is defined in such a way that if $h_{p^*p} : Y_p \to Y_{p^*}$, $\omega_{p^*\chi(p^*)} : X_{\chi(p^*)} \to Y_{p^*}$, then there exists a morphism $h_{\chi(p^*)\chi(p)} : X_{\chi(p)} \to X_{\chi(p^*)}$, and the following diagram is commutative:

$$\begin{tikzcd}
Y_p \arrow[r, "h_{p^*p}"] & Y_{p^*} \\
X_{\chi(p)} \arrow[u, "\omega_{p\chi(p)}"] \arrow[r, "h_{\chi(p^*)\chi(p)}"] & X_{\chi(p^*)} \arrow[u, "\omega_{p^*\chi(p^*)}"']
\end{tikzcd} \qquad (\Phi)$$

(3) if $h_{\chi(p^*)\chi(p)} : X_{\chi(p)} \to X_{\chi(p^*)}$, $\omega_{p\chi(p)} : X_{\chi(p)} \to Y_p$, $\omega_{p^*\chi(p^*)} : X_{\chi(p^*)} \to Y_{p^*}$, then there exists a morphism $h_{p^*p} : Y_p \to Y_{p^*}$ and the diagram (Φ) is commutative.

It follows from condition (3) and the definition of a Hausdorff spectrum that, for example, every diagram

$$\begin{tikzcd}
Y_p \arrow[rr] & & Y_{p^*} \\
& Y_{p'} \arrow[ur] & \\
X_{\chi(p)} \arrow[uu] \arrow[rr] & & X_{\chi(p')} \arrow[ul]
\end{tikzcd} \qquad (\Phi')$$

is commutative.

In particular, if $|\mathfrak{F}| = |\mathfrak{F}^1| = \mathbb{Z}$ ($\mathbb{Z}$ is the set of whole numbers), $\mathfrak{F} = \{|\mathfrak{F}|\}$, $\mathfrak{F}^1 = \{|\mathfrak{F}^1|\}$, then we obtain a mapping of inverse spectra. Moreover, V. P. Palamodov's version of mapping of spectra [1] is a mapping of Hausdorff spectra. In fact, for each $\beta \in \mathbb{Z}$ there exists the largest $\alpha = \alpha(\beta) \in \mathbb{Z}$ such that $(\alpha, \beta) \in \Delta_U$, i.e. the inverse function of $\alpha(\beta)$ from Condition II of Definition 2 in [1] defines a set of morphisms $u_\alpha^\beta : X_{\alpha(\beta)} \to Y_\beta$ of the category K $(\beta \in \mathbb{Z})$ which satisfies Condition I – this corresponds to fulfilling (1) and (2).

Suppose that $\omega_{\mathcal{YX}} : \mathcal{X} \to \mathcal{Y}$ and $\omega_{\mathcal{ZY}} : \mathcal{Y} \to \mathcal{Z}$ are mappings of Hausdorff spectra so that $\omega_{\mathcal{YX}} = \omega(\varphi, \phi, \chi)$, $\omega_{\mathcal{ZY}} = \omega(\varphi', \phi', \chi')$. Let us put $\varphi^* = \varphi' \circ \varphi$, $\phi^* = \phi \circ \phi'$, $\chi^* = \chi \circ \chi'$, so that $\varphi^* : \mathfrak{F} \to \mathfrak{F}^2$, $\phi^* : F^2 \to F$ $(\forall F \in \mathfrak{F})$, $\chi^* : T^2 \to T$ $(\forall T^2 \in F^2)$, setting $\omega_{rs} = \omega_{rp} \circ \omega_{ps}$ whenever morphisms ω_{ps} and ω_{rp} are defined. It is easy to verify that the set of morphisms $\omega_{rs} : X_s \to Z_r$ of the category $\mathcal{G}$ satisfies conditions (1) and (2) for a mapping of Hausdorff spectra. We will call the mapping $i : \mathcal{X} \to \mathcal{X}$, where $\mathcal{X} = \{X_s, \mathfrak{F}, h_{s's}\}$, the *identity mapping* if it is formed by means of all the identity morphisms $\omega_{ss} : X_s \to X_s$ $(s \in |\mathfrak{F}|)$ of the category $\mathcal{G}$; it is clear that i is a left and right identity under composition.

Thus, the set of Hausdorff spectra over $\mathcal{G}$ and their mappings form a category, which (by analogy with [1]) we will denote by $\operatorname{Spect}\mathcal{G}$. We may consider the category $\mathcal{G}$ as a subcategory in $\operatorname{Spect}\mathcal{G}$ – namely, to each object $A \in \mathcal{G}$ we assign the Hausdorff spectrum $\mathcal{A} = \{A, \{A\}, \emptyset\}$. Let $\mathcal{X} = \{X_s, \mathfrak{F}, h_{s's}\} \in \operatorname{Spect}\mathcal{G}$. Then every mapping $\omega_{B\mathcal{X}} : \mathcal{X} \to B$ is given by a set of morphisms $\omega_{Bs} : X_s \to B$, where $\varphi : \mathfrak{F} \to \{B\}$, $\phi_F : \{B\} \to F$ $(\forall F \in \mathfrak{F})$, $\chi_F : \{B\} \to \phi_F(\{B\})$ and $s = \chi_F(\phi_F(\{B\}))$ $(F \in \mathfrak{F})$. Correspondingly, every mapping $\omega_{\mathcal{X}A} : A \to \mathcal{X}$ is given by a set of morphisms $\omega_{sA} : A \to X_s$, where $\varphi' : \{A\} \to \mathfrak{F}$, $\phi : F \to \{A\}$ $(F = \varphi'(\{A\}))$, $\chi : T \to \{A\}$ $(\forall T \in F)$, $s \in |F|$, $F = \varphi'(\{A\})$.

Let $\mathcal{X} = \{X_s, \mathfrak{F}, h_{s's}\}$, $\mathcal{Y} = \{Y_p, \mathfrak{F}^1, h_{p'p}\}$ be objects from $\operatorname{Spect}\mathcal{G}$. We will say that two mappings of Hausdorff spectra $\omega_{\mathcal{YX}} : \mathcal{X} \to \mathcal{Y}$ and $\omega'_{\mathcal{YX}} : \mathcal{X} \to \mathcal{Y}$ are *equivalent* if for any $F \in \mathfrak{F}$ there exists $F^* \in \mathfrak{F}^1$ such that the diagram

$$\begin{tikzcd}
 & Y_p \arrow[dr, "h_{p^*p}"] & \\
X_s \arrow[ur, "\omega_{ps}"] \arrow[dr, "\omega'_{p's}"'] & & Y_{p^*} \\
 & Y_{p'} \arrow[ur, "h_{p^*p'}"'] &
\end{tikzcd}$$

is commutative for any $p^* \in |F^*|$ $(s \in |F|,\ p \in |\varphi(F)|,\ p' \in |\varphi'(F)|)$. The relation introduced is reflexive, i.e. in this case $p, p', p^* \in |F|$, $s = \chi(p)$, $s = \chi(p')$, $s^* = \chi(p^*)$ and the following diagram is commutative because of (Φ):

$$\begin{tikzcd}
 & X_{s^*} \arrow[dr, "\omega_{p^*s^*}"] & \\
X_s \arrow[ur, "h_{s^*s}"] \arrow[r, "\omega_{ps}"] \arrow[dr, "\omega'_{p's}"'] & Y_p \arrow[r, "h_{p^*p}"] & Y_{p^*} \\
 & Y_{p'} \arrow[ur, "h_{p^*p'}"'] &
\end{tikzcd}$$

Specifically, the existence of a morphism h_{s^*s} of the Hausdorff spectrum $\mathcal{X}$ follows from (Φ).

We now establish the transitivity of the relation. Let $\omega_{\mathcal{Y}\mathcal{X}} \sim \omega'_{\mathcal{Y}\mathcal{X}}$ and $\omega'_{\mathcal{Y}\mathcal{X}} \sim \omega''_{\mathcal{Y}\mathcal{X}}$. Then transitivity follows from the directedness of the class $\mathfrak{F}^1$ and the commutativity of the following diagram $(\forall\, \overline{p} \in |\overline{F}|)$:

$$\begin{tikzcd}
 & Y_p \arrow[r, "h_{p^*p}"] & Y_{p^*} \arrow[dr, "h_{\bar{p}p^*}"] & \\
X_s \arrow[ur, "\omega_{ps}"] \arrow[r, "\omega_{p's}"] \arrow[dr, "\omega_{p''s}"'] & Y_{p'} \arrow[ur, "h_{p^*p'}"'] \arrow[dr, "h_{p^{**}p'}"] & & Y_{\bar{p}} \\
 & Y_{p''} \arrow[r, "h_{p^{**}p''}"] & Y_{p^{**}} \arrow[ur, "h_{\bar{p}^{**}}"'] &
\end{tikzcd}$$

Here, $p \in |\varphi(F)|$, $p' \in |\varphi'(F)|$, $p'' \in |\varphi''(F)|$, $p^* \in |F^*|$, $p^{**} \in |F^{**}|$ and $F^* \prec \overline{F}$, $F^{**} \prec \overline{F}$. It is clear that the equivalence relation is preserved under composition.

Thus the set $\mathrm{Hom}(\mathcal{X}, \mathcal{Y})$ is decomposed into equivalence classes; let us now consider a new category $\mathcal{H}$ whose objects are the objects of the category $\mathrm{Spect}\,\mathcal{G}$, while the set $\mathrm{Hom}_{\mathcal{H}}(\mathcal{X}, \mathcal{Y})$ is formed by the equivalence classes of mappings $\omega_{\mathcal{Y}\mathcal{X}} : \mathcal{X} \to \mathcal{Y}$. We will denote these classes by $||\omega_{\mathcal{Y}\mathcal{X}}||$.

Let $\mathcal{G}$ be a semiabelian complete subcategory of the category TG, in which it is possible to construct direct sums and direct products. Then for each Hausdorff spectrum $\mathcal{X} = \{X_s, \mathfrak{F}, h_{s's}\}$ over $\mathcal{G}$ there exists (as already shown) a unique (up to isomorphism) object of the category $\mathcal{G}$, called the H-limit of the Hausdorff spectrum $\mathcal{X}$ and denoted by $\overset{\longleftarrow}{\underset{\longrightarrow}{\lim}}_{\mathfrak{F}}\, h_{s's}X_s$. Moreover, if $\omega_{\mathcal{Y}\mathcal{X}} : \mathcal{X} \to \mathcal{Y}$, then there exists a unique morphism

$$\overline{\omega}_{\mathcal{Y}\mathcal{X}} : \overset{\longleftarrow}{\underset{\longrightarrow}{\lim}}_{\mathfrak{F}}\, h_{s's}X_s \to \overset{\longleftarrow}{\underset{\longrightarrow}{\lim}}_{\mathfrak{F}^1}\, h_{p'p}Y_p$$

of the category $\mathcal{G}$. In fact, let $x \in \overset{\longleftarrow}{\underset{\longrightarrow}{\lim}}_{\mathfrak{F}}\, h_{s's}X_s$, i.e. $x \in \bigcup_{F\in\mathfrak{F}} \bigcap_{T\in F} \psi V_F^T$, where $\psi : \widehat{S} \to S$ is the canonical mapping and

$$V_F^T = \{\alpha \in \prod_F X_s : x_s = \hat{h}_{s\hat{s}} x_{\hat{s}}\,,\ s, \hat{s} \in T\}\,.$$

Then there exists $F \in \mathfrak{F}$ such that $x \in \psi V_F^T$ $(T \in F)$, and, consequently, $x = \psi\alpha_T$, where $\alpha_T = (x_s^T)_F$, $\alpha_T \in V_F^T$, $T \in F$. Therefore by the definition of a mapping of Hausdorff spectra there exist $F^1 \in \mathfrak{F}^1$, $F^1 = \varphi(F)$, $\phi : F^1 \to F$ and $\chi : T^1 \to T$ $(\forall\, T^1 \in F^1)$ which allow us to define a morphism of the category

$$g_{F^1F} : \prod_F X_s \to \prod_{F^1} Y_p\,,$$

where $g_{F^1F} = \{\omega_{p\chi(p)}\}_{p\in|F^1|}$. For each $T^1 \in F^1$ we define an element $\beta_{T^1} \in V_{F^1}^{T^1} \subset \prod_{F^1} Y_p$ such that $\beta_{T^1} = \{\omega_{p\chi(p)} x^T_{\chi(p)}\}_{p\in|F^1|}$, where $T = \phi(T^1)$. Here, given $\hat{h}_{p\hat{p}} : Y_{\hat{p}} \to Y_p$, there exists by (Φ)

$\hat{h}_{\chi(p)\chi(\hat{p})} : X_{\chi(\hat{p})} \to X_{\chi(p)}$, and moreover $\hat{h}_{p\hat{p}}(\omega_{\hat{p}\chi(\hat{p})} x^T_{\chi(\hat{p})}) = \omega_{p\chi(p)} x^T_{\chi(p)}$, where $p, \hat{p} \in T^1$. Now if ψ' is the canonical mapping for the Hausdorff spectrum $\mathcal{Y}$, then by (Φ) we obtain $\psi'\beta_{T^1_1} = \psi'\beta_{T^1_2}$ for arbitrary $T^1_1, T^1_2 \in F^1$. It remains to put $y = \psi'\beta_{T^1}$ $(T^1 \in F^1)$, where $y \in \bigcap_{T^1 \in F^1} \psi' V^{T^1}_{F^1}$, and, consequently, $y \in \underset{\overrightarrow{\mathfrak{F}^1}}{\overleftarrow{\lim}}\, h_{p'p} Y_p$ and $\overline{\omega}_{\mathcal{YX}} x = y$. Additivity and continuity of $\overline{\omega}_{\mathcal{YX}}$ are obvious and come directly from the definition of the H-limit of a Hausdorff spectrum, therefore $\overline{\omega}_{\mathcal{YX}}$ is a morphism of the category TG. We will employ the notation $H(\omega_{\mathcal{YX}}) = \overline{\omega}_{\mathcal{YX}}$.

It is clear that H translates the identity mapping into the identity and a composition of mappings into a composition. Therefore H is a covariant functor from the category $\mathrm{Spect}\,\mathcal{G}$ into the category $\mathcal{G}$. Moreover, we have the following result:

Proposition 1. *Let $H : \mathrm{Spect}\,\mathcal{G} \to \mathcal{G}$. Then H can be extended to the category $\mathcal{H}$ and is additive on it.*

Proof. We show first of all that $\mathrm{Hom}_{\mathcal{H}}(\mathcal{X}, \mathcal{Y})$ is an abelian group. Let $\omega_{\mathcal{YX}} : \mathcal{X} \to \mathcal{Y}$, $\omega'_{\mathcal{YX}} : \mathcal{X} \to \mathcal{Y}$, where $\mathcal{X}, \mathcal{Y} \in \mathrm{Spect}\,\mathcal{G}$, $\omega_{\mathcal{YX}} = \omega(\varphi, \phi, \chi)$, $\omega'_{\mathcal{YX}} = \omega'(\varphi', \phi', \chi')$. For each $F \in \mathfrak{F}$ we can find $F^* \in \mathfrak{F}^1$ such that $\varphi(F) \prec F^*$ and $\varphi'(F) \prec F^*$. Let us construct mappings of Hausdorff spectra $\overline{\omega}^{1,2}_{\mathcal{YX}} : \mathcal{X} \to \mathcal{Y}$ so that $\omega_{\mathcal{YX}} \sim \overline{\omega}^1_{\mathcal{YX}}$ and $\omega'_{\mathcal{YX}} \sim \overline{\omega}^2_{\mathcal{YX}}$, $\bigcup_{F^*} \overline{\chi}^1(p) = \bigcup_{F^*} \overline{\chi}^2(p)$. In fact, for $p \in |F^1|$ there exists $s_p \in |F|$ such that $\hat{h}_{\chi(p)s_p} : X_{s_p} \to X_{\chi(p)}$, $\hat{h}_{\chi'(p)s_p} : X_{s_p} \to X_{\chi'(p)}$ and moreover, if $p \in T^*$, $\omega_{FF^*} : F^* \to F$, $\omega_{F'F^*} : F^* \to F'$, then $s_p \in T$, where $T = \overline{\phi}(T^*)$, $T \supset \phi[\omega_{FF^*}(T^*)]$, $T \supset \phi'[\omega_{F'F^*}(T^*)]$. Putting $\overline{\chi}^1(p) = s_p$ and $\overline{\chi}^2(p) = s_p$ in this case, we obtain the necessary identity. Now if we put $\overline{\varphi}(F) = F^*$, then the mappings of Hausdorff spectra $\overline{\omega}^1_{\mathcal{YX}} = \omega(\overline{\varphi}, \overline{\phi}, \overline{\chi}^1)$, $\overline{\omega}^2_{\mathcal{YX}} = \omega(\overline{\varphi}, \overline{\phi}, \overline{\chi}^2)$ are equivalent to $\omega_{\mathcal{YX}}$ and $\omega'_{\mathcal{YX}}$ respectively by (Φ). Therefore we define $||\omega_{\mathcal{YX}}|| + ||\omega'_{\mathcal{YX}}||$ to be the element of $\mathrm{Hom}_{\mathcal{H}}(\mathcal{X}, \mathcal{Y})$ containing

$$\{\omega_{p\overline{\chi}^1(p)} + \omega_{p\overline{\chi}^2(p)}\}_{p \in |F^*|} \quad (F \in \mathfrak{F},\ F^* = \overline{\varphi}(F)).$$

Clearly, this class does not depend on the choice of representatives $\omega_{\mathcal{YX}}$, $\omega'_{\mathcal{YX}}$ in their equivalence classes. The operation of addition which has been introduced converts $\mathrm{Hom}_{\mathcal{H}}(\mathcal{X}, \mathcal{Y})$ into an abelian group. Now the extension of the functor H to the category $\mathcal{H}$ and its additivity there are obvious. The proposition is proved.

We will reserve the notation $H = \mathrm{Haus}$ for the case $\mathcal{G} = TLC$.

We introduce a semiabelian structure on the category $\mathcal{H}$. For any objects $\mathcal{X}, \mathcal{Y}, \mathcal{Z} \in \mathcal{H}$ the law of composition defines a bilinear mapping

$$\mathrm{Hom}_{\mathcal{H}}(\mathcal{X}, \mathcal{Y}) \times \mathrm{Hom}_{\mathcal{H}}(\mathcal{Y}, \mathcal{Z}) \to \mathrm{Hom}_{\mathcal{H}}(\mathcal{X}, \mathcal{Z}).$$

Thus $\mathcal{H}$ is an additive category.

Proposition 2. (See [1].) *The category $\mathcal{H}$ is semiabelian.*

Proof. Let $||\omega_{\mathcal{YX}}|| : \mathcal{X} \to \mathcal{Y}$, where $\mathcal{X}, \mathcal{Y}$ are Hausdorff spectra over $\mathcal{G}$. We will construct for a morphism $||\omega_{\mathcal{YX}}||$ of the category $\mathcal{H}$ its kernel and cokernel. We choose in the class $||\omega_{\mathcal{YX}}||$ some element $\omega_{\mathcal{YX}} \in \mathrm{Spect}\,\mathcal{G}$ so that $\omega_{\mathcal{YX}} = \omega(\varphi, \phi, \chi)$. Now for each $s \in |F|$, where $F \in \mathfrak{F}$, let us consider an object $N_s \in \mathcal{G}$, $N_s \subset X_s$, provided with the topology induced from X_s, and such that $N_s = \ker \omega_{ps}$ for $s = \chi(p)$ $(p \in |\varphi(F)|)$. By (Φ) the restriction $n_{s's}$ of the morphism $h_{s's}$ translates N_s into $N_{s'}$, therefore the family $\mathcal{N} = \{N_s, \mathfrak{F}, n_{s's}\}$ is a Hausdorff subspectrum of the Hausdorff spectrum $\mathcal{X} = \{X_s, \mathfrak{F}, h_{s's}\}$. We will show that the identity embedding $i_{\mathcal{XN}} : \mathcal{N} \to \mathcal{X}$

is the kernel of $\omega_{\mathcal{Y}\mathcal{X}}$. For this it is enough to establish that for any morphism $m_{\mathcal{X}\mathcal{Z}} : \mathcal{Z} \to \mathcal{X}$ of the category $\operatorname{Spect}\mathcal{G}$ such that $\omega_{\mathcal{Y}\mathcal{X}} \circ m_{\mathcal{X}\mathcal{Z}} = 0$,

$$\mathcal{Z} \in \operatorname{Spect}\mathcal{G}\,, \quad \mathcal{Z} = \{Z_t, \mathfrak{F}^\circ, h_{t't}\}$$

there exists a morphism $n_{\mathcal{N}\mathcal{Z}} : \mathcal{Z} \to \mathcal{N}$ of the category $\operatorname{Spect}\mathcal{G}$ such that the following diagram is commutative:

$$\begin{tikzcd} \mathcal{N} \arrow[dr, "i_{\mathcal{X}\mathcal{N}}"] & & \\ & \mathcal{X} \arrow[r, "\omega_{\mathcal{Y}\mathcal{X}}"] & \mathcal{Y} \\ \mathcal{Z} \arrow[uu, "n_{\mathcal{N}\mathcal{Z}}"] \arrow[ur, "m_{\mathcal{X}\mathcal{Z}}"'] & & \end{tikzcd} \tag{N}$$

(Here the zero mapping of spectra signifies that for $\omega_{\mathcal{Y}\mathcal{X}} = 0 = \omega_{\mathcal{Y}\mathcal{X}} \circ m_{\mathcal{X}\mathcal{Z}}$ its component morphisms $\omega_{pt} = \omega_{ps} \circ \omega_{st}$ are such that $\omega_{pt}(Z_t) = 0$.)

At the same time it is clear that, if for $\omega'_{\mathcal{Y}\mathcal{X}} \sim \omega_{\mathcal{Y}\mathcal{X}}$, $m'_{\mathcal{X}\mathcal{Z}} \sim m_{\mathcal{X}\mathcal{Z}}$ such that $\omega'_{\mathcal{Y}\mathcal{X}} \circ m'_{\mathcal{X}\mathcal{Z}} = 0'$, where $0' \sim 0$, there also exist $n'_{\mathcal{N}\mathcal{Z}} \in \operatorname{Spect}\mathcal{G}$ and $i'_{\mathcal{X}\mathcal{N}} \sim i_{\mathcal{X}\mathcal{N}}$ such that the diagram

$$\begin{tikzcd} \mathcal{N} \arrow[dr, "i'_{\mathcal{X}\mathcal{N}}"] & & \\ & \mathcal{X} \arrow[r, "\omega'_{\mathcal{Y}\mathcal{X}}"] & \mathcal{Y} \\ \mathcal{Z} \arrow[uu, "n'_{\mathcal{N}\mathcal{Z}}"] \arrow[ur, "m'_{\mathcal{X}\mathcal{Z}}"'] & & \end{tikzcd}$$

is commutative, then $n'_{\mathcal{N}\mathcal{Z}} \sim n_{\mathcal{N}\mathcal{Z}}$. Therefore, if diagram (N) applies, each morphism $||\omega_{\mathcal{Y}\mathcal{X}}||$ of the category $\mathcal{H}$ such that $||\omega_{\mathcal{Y}\mathcal{X}}|| \circ ||m_{\mathcal{X}\mathcal{Z}}|| = 0$, where $||m_{\mathcal{X}\mathcal{Z}}|| : \mathcal{Z} \to \mathcal{X}$, and $i_{\mathcal{X}\mathcal{N}} \in ||i_{\mathcal{X}\mathcal{N}}||$, has kernel $||i_{\mathcal{X}\mathcal{N}}||$ such that there exists $||n_{\mathcal{N}\mathcal{Z}}||$ with commutative diagram

$$\begin{tikzcd} \mathcal{N} \arrow[dr, "||i_{\mathcal{X}\mathcal{N}}||"] & & \\ & \mathcal{X} \arrow[r, "||\omega_{\mathcal{Y}\mathcal{X}}||"] & \mathcal{Y} \\ \mathcal{Z} \arrow[uu, "||n_{\mathcal{N}\mathcal{Z}}||"] \arrow[ur, "||m_{\mathcal{X}\mathcal{Z}}||"'] & & \end{tikzcd}$$

Thus, for the existence of the kernel of the morphism $||\omega_{\mathcal{Y}\mathcal{X}}||$ it is enough to establish the existence of $n_{\mathcal{N}\mathcal{Z}} : \mathcal{Z} \to \mathcal{N}$ and the commutativity of diagram (N).

If the mapping of spectra is $m_{\mathcal{X}\mathcal{Z}} = m(\varphi^\circ, \phi^\circ, \chi^\circ)$, then taking into account the fact that $\operatorname{Im}\omega_{s\chi^\circ(s)} \subset N_s$ $(s \in |\phi^\circ(F^\circ)|,\ F^\circ \in \mathfrak{F}^\circ)$ by assumption, we can construct a mapping of Hausdorff spectra $n_{\mathcal{N}\mathcal{Z}} : \mathcal{Z} \to \mathcal{N}$, where $n_{\mathcal{N}\mathcal{Z}} = n(\varphi^\circ, \phi^\circ, \chi^\circ)$, so that its constituent morphisms $\widehat{\omega}_{s\chi^\circ(s)} : Z_{\chi^\circ(s)} \to N_s$ are restrictions of the morphisms $\omega_{s\chi^\circ(s)}$. Commutativity of the diagram is obvious.

Now we will construct the cokernel of the morphism $||\omega_{\mathcal{Y}\mathcal{X}}||$; let $\omega_{\mathcal{Y}\mathcal{X}} \in ||\omega_{\mathcal{Y}\mathcal{X}}||$. For each $p \in |\varphi(F)|$ $(F \in \mathfrak{F})$ let us consider the factor group $R_p = Y_p/\operatorname{Im}\omega_{p\chi(p)}$ with the topology induced from Y_p. It is clear that because of (Φ) the subgroups $\operatorname{Im}\omega_{p\chi(p)}$ form a Hausdorff spectrum, therefore the factor groups R_p $(p \in |\varphi(F)|)$ also form a Hausdorff spectrum; let

$$\mathcal{R} = \{R_p, \mathfrak{F}_0^1, h_{p'p}\}\,, \quad \mathcal{Y}_0 = \{Y_p, \mathfrak{F}_0^1, h_{p'p}\}\,,$$

where $\mathfrak{F}_0^1 = \mathfrak{F}^1|_{\varphi(\mathfrak{F})}$ (without loss of generality we may assume that $\varphi(\mathfrak{F}) = \mathfrak{F}^1$). Let us denote by $\omega_{\mathcal{R}\mathcal{Y}} : \mathcal{Y} \to \mathcal{R}$ the canonical mapping of Hausdorff spectra; we will show that $||\omega_{\mathcal{R}\mathcal{Y}}||$ is the cokernal of the morphism $||\omega_{\mathcal{Y}\mathcal{X}}||$. For this it follows that we have to establish that, for any morphism $m_{\mathcal{Z}\mathcal{Y}} : \mathcal{Y} \to \mathcal{Z}$ of the category $\operatorname{Spect}\mathcal{G}$ such that $m_{\mathcal{Z}\mathcal{Y}} \circ \omega_{\mathcal{Y}\mathcal{X}} = 0$, there exists a morphism $n_{\mathcal{Z}\mathcal{R}} : \mathcal{R} \to \mathcal{Z}$ of the category $\operatorname{Spect}\mathcal{G}$ such that the following diagram is commutative:

$$\begin{tikzcd} & & \mathcal{Z} \\ \mathcal{X} \arrow[r, "\omega_{\mathcal{Y}\mathcal{X}}"] & \mathcal{Y} \arrow[ur, "m_{\mathcal{Z}\mathcal{Y}}"] \arrow[dr, "\omega_{\mathcal{X}\mathcal{Y}}"'] & \\ & & \mathcal{R} \arrow[uu, "n_{\mathcal{Z}\mathcal{R}}"'] \end{tikzcd} \qquad (K)$$

If $m_{\mathcal{Z}\mathcal{Y}} = m(\widehat{\varphi}, \widehat{\phi}, \widehat{\chi})$, then $n_{\mathcal{Z}\mathcal{R}} = n(\widehat{\varphi}, \widehat{\phi}, \widehat{\chi})$, and since $\omega_{z\widehat{\chi}(z)}(Y_{\widehat{\chi}(z)}) = 0$ for all $z \in |\widehat{\varphi}(F^1)|$ $(F^1 \in \mathfrak{F})$, then $\operatorname{Im}\omega_{\widehat{\chi}(z)\chi(\widehat{\chi}(z))} \subset N_{\widehat{\chi}(z)}$, and, consequently, because the category $\mathcal{G}$ is semiabelian there exists a morphism $\widehat{\omega}_{z\widehat{\chi}(z)} : R_{\widehat{\chi}(z)} \to Z_t$ such that the following diagram is commutative:

$$\begin{tikzcd} & Z_z \\ Y_{\hat{\chi}(z)} \arrow[ur, "\omega_{z\hat{\chi}(z)}"] \arrow[dr, "\omega_{\hat{\chi}(z)\hat{\chi}(z)}"'] & \\ & R_{\hat{\chi}(z)} \arrow[uu, "\hat{\omega}_{z\hat{\chi}(z)}"'] \end{tikzcd}$$

Thus, as is not difficult to see, the set of morphisms $\widehat{\omega}_{z\widehat{\chi}(z)}$ defines a mapping of Hausdorff spectra in such a way that diagram (K) is commutative. The proposition is proved.

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