ON ROOT-CLASS RESIDUALITY OF GENERALIZED FREE PRODUCTS

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ABSTRACT. We prove root-class residuality of free product of root-class residual groups. A sufficient condition for root-class residuality of generalized free product G of groups A and B amalgamating subgroups H and K under an isomorphism φ is derived. For the particular case when $A = B$, $H = K$ and φ is the identical mapping, it is shown that group G is root-class residual if and only if A is root-class residual and subgroup H of A is root-class separable. These results are extended to generalized free product of infinitely many groups amalgamating a common subgroup.

1. Introduction

Let $\mathcal{K}$ be an abstract class of groups. Suppose $\mathcal{K}$ contains at least a non trivial group. Then $\mathcal{K}$ is called a root-class if the following conditions are satisfied:

1. If $A \in \mathcal{K}$ and $B \leqslant A$, then $B \in \mathcal{K}$.
2. If $A \in \mathcal{K}$ and $B \in \mathcal{K}$, then $A \times B \in \mathcal{K}$.
3. If $1 \leqslant C \leqslant B \leqslant A$ is a subnormal sequence and A/B, $B/C \in \mathcal{K}$, then there exists a normal subgroup D in group A such that $D \leqslant C$ and $A/D \in \mathcal{K}$. See for example [4], p. 428 for details about this definition.

In this paper, we study root-class residuality of generalized free products.

We recall that a group G is root-class residual (or $\mathcal{K}$-residual), for a root-class $\mathcal{K}$ if, for every $1 \neq g \in G$, there exists a homomorphism φ of group G onto a group of root-class $\mathcal{K}$ such that $g\varphi \neq 1$. Equivalently, group G is $\mathcal{K}$-residual if, for every $1 \neq g \in G$, there exists a normal subgroup N of G such that $G/N \in \mathcal{K}$ and $g \notin N$. The most investigated residual properties of groups are finite groups residuality (residual finiteness), p-finite groups residuality and soluble groups residuality. All these three classes of groups are root-classes. Therefore results about root-class residuality have sufficiently enough general character.

In [4] (p. 429) the following result obtained by Gruenberg is given:

Free product of root-class residual groups is root-class residual if and only if every free group is root-class residual.

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The following theorem shown in item 2 of this paper asserts that the given above necessary and sufficient condition is satisfied for every root-class:

Theorem 1. *Every free group is root-class residual, for every root-class.*

So, Gruenberg's result can be reformulated as follows:

Theorem 2. *Free product of root-class residual groups is root-class residual.*

From theorem 2 and H. Neumann's theorem (see [3], p. 122), the following result is easily established:

Theorem 3. *Let $\mathcal{K}$ be a root-class. The generalized free product $G = A \underset{H}{\star} B$ of groups A and B amalgamating subgroup H is $\mathcal{K}$-residual if groups A and B are $\mathcal{K}$-residual and there exists a homomorphism φ of group G onto a group of root-class $\mathcal{K}$ such that φ is one-to-one on H.*

We remark that theorem 2 can be considered as a particular case of theorem 3.

We also see that, if the amalgamated subgroup H is finite, then the formulated above sufficient condition of root-class residuality of group G will be as well necessary.

Another restriction permitting to obtain simple criteria of root-class residuality of generalized free product of groups A and B amalgamating subgroup H is the equality of the free factors A and B.

More precisely, let G be the generalized free product of groups A and B amalgamating subgroups H and K under an isomorphism φ. If $A = B$, $H = K$ and φ is the identical mapping, we denote group G by $G = A \underset{H}{\star} A$. Then for such group we prove the following criterium:

Theorem 4. *Let $\mathcal{K}$ be a root-class. Group $G = A \underset{H}{\star} A$ is $\mathcal{K}$-residual if and only if group A is $\mathcal{K}$-residual and subgroup H of A is $\mathcal{K}$-separable.*

In [2] the above result is obtained for the particular case of the class of all finite p-groups. We recall that subgroup H of a group A is root-class separable (or $\mathcal{K}$-separable), for a root-class $\mathcal{K}$ if, for any element a of A where $a \notin H$, there exists a homomorphism φ of group A onto a group of root-class $\mathcal{K}$ such that $a\varphi \notin H\varphi$. This means that, for any $a \in A \setminus H$, there exists a normal subgroup N of A such that $A/N \in \mathcal{K}$ and $a \notin NH$.

Further, the generalized free product of infinitely many groups amalgamating subgroup is introduced, for example, in [5]. We extend theorems 3 and 4 above to generalized free products of a family $(G_\lambda)_{\lambda \in \Lambda}$ of groups amalgamating a common subgroup H (theorems 5 and 6). The set Λ of indices can be infinite.

2. Proof of theorems 1-4

We first prove the following lemma.

Lemma. *Let $\mathcal{K}$ be a root-class of groups. Then*

1. If a group G has a subnormal sequence with factors belonging to class $\mathcal{K}$, then $G \in \mathcal{K}$.

2. If $F \trianglelefteq G$, $G/F \in \mathcal{K}$ and F is $\mathcal{K}$-residual, then group G is also $\mathcal{K}$-residual.
3. If $A \trianglelefteq G$, $B \trianglelefteq G$, $G/A \in \mathcal{K}$ and $G/B \in \mathcal{K}$, then $G/(A \cap B) \in \mathcal{K}$.

In fact, from the definition of root-class, it follows that root-class is closed under any extension. So the first property of lemma is satisfied. The second and third properties are also easily verified by the definition of root-class.

For the proof of theorem 1, let's remark that every root-class $\mathcal{K}$ contains a nontrivial cyclic group (property 1 of the definition of root-class). If $\mathcal{K}$ contains an infinite cyclic group then, by lemma, $\mathcal{K}$ contains any group possessing subnormal sequence with infinite cyclic factors; and thus, all finitely generated nilpotent torsion-free groups belong to class $\mathcal{K}$. If $\mathcal{K}$ contains a finite nontrivial cyclic group, then $\mathcal{K}$ contains group of prime order p and consequently, by lemma, $\mathcal{K}$ contains every group possessing subnormal sequence with factors of order p; hence all finite p-groups belong to $\mathcal{K}$. So, every root-class contains all finitely generated nilpotent torsion-free groups or all finite p-groups, for some prime p. But free groups are residually finitely generated nilpotent torsion-free groups and also are residually finite p-groups (see [4], p. 347 and [1], p. 121). Therefore, free groups are $\mathcal{K}$-residual, for every root-class $\mathcal{K}$ and this ends the proof of theorem 1.

Now, theorem 2 directly follows from theorem 1 and Gruenberg's result formulated above.

We prove theorem 3.
Let $\mathcal{K}$ be a root-class. Let $G = A \underset{H}{*} B$ be the generalized free product of groups A and B amalgamating subgroup H and assume that groups A and B are $\mathcal{K}$-residual. Suppose there exists a homomorphism σ of G onto a group of class $\mathcal{K}$ such that σ is one-to-one on H. Let's denote by N the kernel of the homomorphism σ. Then $G/N \in \mathcal{K}$ and $N \cap H = 1$. By H. Neumann's theorem ([3], p. 122), N is the the free product of a free group F and some subgroups of group G of the form

$$g^{-1}Ag \cap N, \quad g^{-1}Bg \cap N, \qquad (*)$$

where $g \in G$. The subgroups of the form $(*)$ are $\mathcal{K}$-residual since are groups A and B. By theorem 1, free group F is also $\mathcal{K}$-residual. Thus N is a free product of $\mathcal{K}$-residual groups. Therefore, by theorem 2, N is $\mathcal{K}$-residual. Moreover, since $G/N \in \mathcal{K}$, by property 2 of lemma, it follows that group G is $\mathcal{K}$-residual. Hence, theorem 3 is proven.

We prove theorem 4.
Let $\mathcal{K}$ be a root-class. Let $G = A \underset{H}{*} A$. For every normal subgroup N of group A one can define the generalized free product

$$G_N = A/N \underset{HN/N}{*} A/N$$

and the homomorphism $\varepsilon_N : G \longrightarrow G_N$, extending the canonical homomorphism $A \longrightarrow A/N$. It is evident that group G_N is an extension of a free group by group A/N. So, if

A/N belongs to root-class $\mathcal{K}$ then, by lemma and theorem 1, G_N is $\mathcal{K}$-residual. Thus, to prove $\mathcal{K}$-residuality of group G, it is enough to show that G is residually a group of kind G_N, where $A/N \in \mathcal{K}$.

Suppose group A is $\mathcal{K}$-residual and subgroup H of A is $\mathcal{K}$-separable. Let $1 \neq g \in G$. Assume that element g has a reduced form $g = a_1 \cdots a_s$. Two cases arise:

1. $s > 1$. In this case $a_i \in A \setminus H$ for all $i = 1, \ldots, s$. From $\mathcal{K}$-separability of H, it follows that, for every $i = 1, \cdots, s$, there exits a normal subgroup N_i of group A such that $A/N_i \in \mathcal{K}$ and $a_i \notin HN_i$. Let $N = N_1 \cap \cdots \cap N_s$. By lemma, $A/N \in \mathcal{K}$ and, it is clear that, for every $i = 1, \cdots, s$, $a_i \notin HN$ i.e. $a_iN \notin HN/N$. So, for every $i = 1, \cdots, s$, $a_i\varepsilon_N \notin H\varepsilon_N$. Therefore the form

$$g\varepsilon_N = a_1\varepsilon_N \cdots a_s\varepsilon_N$$

is reduced and has length $s > 1$. Consequently $g\varepsilon_N \neq 1$.

2. $s = 1$ i.e. $g \in A$. Since group A is $\mathcal{K}$-residual, then there exists a normal subgroup N of A such that $A/N \in \mathcal{K}$ and $g \notin N$, i.e. $gN \neq N$. Hence $g\varepsilon_N \neq 1$.

Thus, for any element $g \neq 1$, there exists a normal subgroup N in group A, such that $A/N \in \mathcal{K}$ and the homomorphism $\varepsilon_N : G \longrightarrow G_N$ mapped element g to a non identity element. Hence group G is residually a group G_N where $A/N \in \mathcal{K}$. Therefore G is $\mathcal{K}$-residual.

Conversely, suppose group G is $\mathcal{K}$-residual. Evidently, its subgroup A has the same property. We prove that H is a $\mathcal{K}$-separable subgroup of group A. Let γ be an automorphism of group G canonically permuting the free factors. Let $a \in A \setminus H$. Then $a\gamma \neq a$. Since G is $\mathcal{K}$-residual, then there exists a normal subgroup N of G such that $G/N \in \mathcal{K}$ and $aN \neq a\gamma N$. Let $M = N \cap N\gamma$. Then

$$M\gamma = N\gamma \cap N\gamma^2 = N\gamma \cap N = M.$$

Consequently, in the factor group G/M, we consider the automorphism $\overline{\gamma}$, induced by γ. Since $aN \neq a\gamma N$ and $M \leqslant N$, then $aM \neq a\gamma M$. On the other hand, $a\gamma M = (aM)\overline{\gamma}$. Thus $aM \neq (aM)\overline{\gamma}$. Since γ acts identically on H, then $\overline{\gamma}$ also acts identically on HM/M. So then, since $aM \neq (aM)\overline{\gamma}$, it follows that $aM \notin HM/M$ i.e. $a\varepsilon \notin H\varepsilon$, where ε is the canonical homomorphism of group G onto G/M. Hence, $G/M \in \mathcal{K}$ and $\mathcal{K}$-separability of subgroup H in group A is demonstrated.

We remark that, the necessary condition for theorem 4 holds even at more gentle restriction on class $\mathcal{K}$, namely when $\mathcal{K}$ satisfies only properties 1 and 2 of the definition of root-class.

3. Generalization

Let $(G_\lambda)_{\lambda\in\Lambda}$ be a family of groups, where set Λ can be infinite. Let $H_\lambda \leqslant G_\lambda$, for every $\lambda \in \Lambda$. Suppose also that, for every $\lambda, \mu \in \Lambda$, there exists an isomorphism $\varphi_{\lambda\mu} : H_\lambda \to H_\mu$ such that, for all $\lambda, \mu, \nu \in \Lambda$, the following conditions are satisfied: $\varphi_{\lambda\lambda} = id_{H_\lambda}$, $\varphi_{\lambda\mu}^{-1} = \varphi_{\mu\lambda}$, $\varphi_{\lambda\mu}\varphi_{\mu\nu} = \varphi_{\lambda\nu}$. Let now

$$G = (\mathop{*}_{\lambda\in\Lambda} G_\lambda;\ h\varphi_{\lambda\mu} = h\ (h \in H_\lambda,\ \lambda, \mu \in \Lambda))$$

be the group generated by groups G_λ $(\lambda \in \Lambda)$ and defined by all the relators of groups G_λ $(\lambda \in \Lambda)$ and moreover by all possible relations: $h\varphi_{\lambda\mu} = h$, where $h \in H_\lambda$, $\lambda, \mu \in \Lambda$. It is evident that every G_λ can be canonically embedded in group G and if we consider $G_\lambda \leqslant G$ then, for all different $\lambda, \mu \in \Lambda$,

$$G_\lambda \cap G_\mu = H_\lambda = H_\mu = H,$$

where H denotes the common subgroups H_λ and thus is a subgroup of any group G_λ $(\lambda \in \Lambda)$. Then G is the generalized free product of groups G_λ $(\lambda \in \Lambda)$ amalgamating subgroup H. One can consider, when necessary, that $G_\lambda \leqslant G$, for all $\lambda \in \Lambda$.

For this construction, we have the following results.

Theorem 5. *Let $\mathcal{K}$ be a root-class. The generalized free product G of groups G_λ $(\lambda \in \Lambda)$ amalgamating subgroup H is $\mathcal{K}$-residual if every group G_λ is $\mathcal{K}$-residual and there exists a homomorphism φ of G onto a group of root-class $\mathcal{K}$ such that φ is one-to-one on H.*

The proof is similar to the proof of theorem 3.

Suppose now that, for all $\lambda \in \Lambda$, $G_\lambda = A$. Then the generalized free product G of group G_λ amalgamating subgroup H is called the generalized free power of group A over subgroup H with index Λ. For such group we have the following criterium which generalizes theorem 4 and the proof is just a repetition of the proof of theorem 4:

Theorem 6. *Let $\mathcal{K}$ be a root-class. The generalized free power G of group A over subgroup H with index Λ is $\mathcal{K}$-residual if and only if group A is $\mathcal{K}$-residual and subgroup H is $\mathcal{K}$-separable in A.*

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